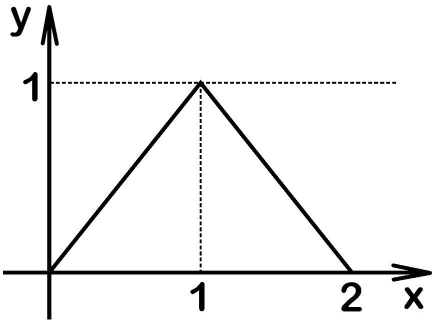
Prvi parcijalni iz Analize III, 19.11.2014.
ispit pisati isključivo hemijskom olovkom



1. Dio grafika f-je $y = f(x)$ je prikazan na slici lijevo. Datu funkciju pretvoriti u Furijer-ov red samo po sin-inusima. Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

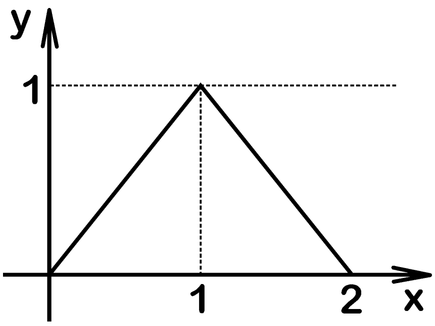
2. Odrediti za koje realane brojeve α dati limes $\lim\limits_{(x,y,z)\to(0,0,0)} z(x^2+y^2+z^2)^{\alpha}$ postoji.

3. Ako je $z = z(x,y)$ i $x + y + z = f(x^2+y^2+z^2)$ izračunati

$$(y-z)\frac{\partial z}{\partial x} + (z-x)\frac{\partial z}{\partial y} + y.$$

4. Izračunati $\iint\limits_D \sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y$ gdje je $D = \{(x,y) \in \mathbb{R}^2 : x \le x^2+y^2 \le 2x\}$.

Prvi parcijalni iz Analize III, 19.11.2014.
ispit pisati isključivo hemijskom olovkom



1. Dio grafika f-je $y = f(x)$ je prikazan na slici lijevo. Datu funkciju pretvoriti u Furijer-ov red samo po sin-inusima. Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

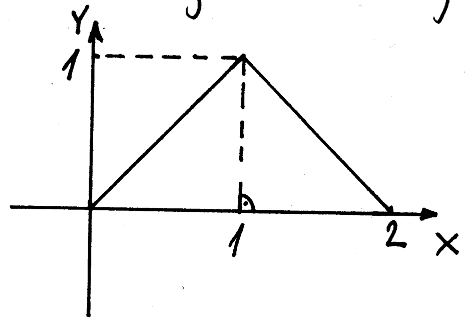
2. Odrediti za koje realane brojeve α dati limes $\lim\limits_{(x,y,z)\to(0,0,0)} z(x^2+y^2+z^2)^{\alpha}$ postoji.

3. Ako je $z = z(x,y)$ i $x + y + z = f(x^2+y^2+z^2)$ izračunati

$$(y-z)\frac{\partial z}{\partial x} + (z-x)\frac{\partial z}{\partial y} + y.$$

4. Izračunati $\iint\limits_D \sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y$ gdje je $D = \{(x,y) \in \mathbb{R}^2 : x \le x^2+y^2 \le 2x\}$.

(#) Dio grafika f-je $y=f(x)$ je prikazan na slici.



Datu f-ju razviti u Furijer-ov red samo po sin-usima. Dobijeni rezultat dobiti za sumiranje reda

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}.$$

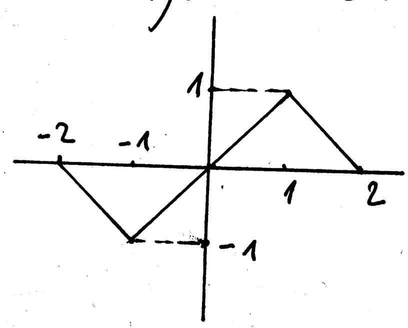
Rj: Furijerov red za $y=f(x)$ na (a,b)

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{b-a}+b_n\sin\frac{2n\pi x}{b-a}\right)$$

gdje se Furijerovi koeficijenti a_n, b_n, računaju po formuli

$$a_0=\frac{2}{b-a}\int_a^b f(x)\,dx,\quad a_n=\frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx,\quad b_n=\frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx.$$

Da bi f-ju razvili u Furijer-ov red samo po sinusima trebamo naštimati da je $a_n=0$ tj. da je $f(x)$ neparna f-ja i naš interval treba biti simetričan. Drugim riječima f-ju koju razvijamo u Furijerov red na intervalu $[-2,2]$ će izgledati ovako:



$(-2;0)$ $(-1;-1)$ $\quad \frac{x+2}{1}=\frac{y-0}{-1}$

$y=-x-2$

$(1,1)$ $(2,0)$ $\quad \frac{x-1}{1}=\frac{y-1}{-1}$

$y-1=-x+1$

$y=-x+2$

$$f(x)=\begin{cases}-x-2, & x\in[-2,-1)\\ x, & x\in[-1,1)\\ -x+2, & x\in[1,2)\end{cases}$$

$$\overset{a}{[-2},\overset{b}{2]}\Rightarrow b-a=4,\quad \frac{2n\pi x}{b-a}=\frac{2n\pi x}{4}=\frac{n\pi x}{2},\quad \frac{2}{b-a}=\frac{2}{4}=\frac{1}{2}$$

$$b_n = \frac{1}{2}\int_{-2}^{2} \underbrace{\underbrace{f(x)}_{\text{nep.}}\ \underbrace{\sin\frac{n\pi x}{2}}_{\text{nep}}}_{\text{parna}}\, dx = \frac{1}{2}\cdot 2\int_{0}^{2} f(x)\sin\frac{n\pi x}{2}\, dx =$$

$$= \int_{0}^{1} x\sin\frac{n\pi x}{2}\, dx + \int_{1}^{2} (-x+2)\cdot\sin\frac{nx\pi}{2}\, dx$$

$$\int_{0}^{1} x\sin\frac{n\pi x}{2}\, dx = \left| \begin{array}{lll} u = x & dv = \sin\frac{n\pi x}{2}\, dx & d\left(\frac{n\pi x}{2}\right) = \frac{n\pi}{2}\, dx \\ du = dx & v = -\frac{2}{n\pi}\cos\frac{n\pi x}{2} & \end{array} \right| =$$

$$= -\frac{2}{n\pi}\, x\cos\frac{n\pi x}{2}\Big|_0^1 + \underbrace{\frac{2}{n\pi}\int_0^1 \cos\frac{n\pi x}{2}\, dx}_{\frac{2}{n\pi}\sin\frac{n\pi x}{2}\big|_0^1} = -\frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\int_1^2 (-x+2)\sin\frac{nx\pi}{2}\, dx = \left| \begin{array}{ll} u = -x+2 & dv = \sin\frac{n\pi x}{2} \\ du = -dx & v = \frac{-2}{n\pi}\cos\frac{n\pi x}{2} \end{array} \right| = \frac{-2}{n\pi}(-x+2)\cos\frac{n\pi x}{2}\Big|_1^2 - \frac{2}{n\pi}\int_1^2 \cos\frac{n\pi x}{2}\, dx$$

$$= \frac{2}{n\pi}\cos\frac{n\pi}{2} - \frac{2}{n\pi}\cdot\frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_1^2 = \frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi}\sin\frac{n\pi}{2}$$

$$f(x) \sim \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2}}{n^2}\sin\frac{n\pi x}{2} = \frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{\sin\frac{(2k-1)\pi}{2}}{(2k-1)^2}\sin\frac{(2k-1)\pi x}{2}$$

$$= \frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^2}\sin\frac{(2k-1)\pi x}{2}$$ traženi Furijer-ov red

za $x = 1$ $\quad f(x) = 1 = \frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^2}\cdot(-1)^{k+1}$

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$$

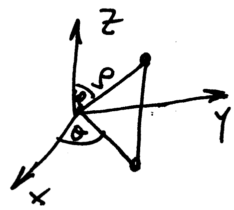
(#) Odrediti realan broj α tako da

$$\lim_{(x,y,z)\to(0,0,0)} z\left(x^2+y^2+z^2\right)^{\alpha}$$

postoji.

Rj. U rješavanju zadatka uvešćemo sferne koordinate

$$x = \rho \sin\varphi \cos\theta$$
$$y = \rho \sin\varphi \sin\theta$$
$$z = \rho \cos\varphi$$



$$x^2+y^2+z^2 = \ldots = \rho^2$$

Tada kad $(x,y,z)\to(0,0,0)$ mi imamo $\rho\to 0$

$$\lim_{(x,y,z)\to(0,0,0)} z\left(x^2+y^2+z^2\right)^{\alpha} = \left|\begin{array}{c}\text{uvodimo}\\ \text{sferne}\\ \text{koordinate}\end{array}\right| = \lim_{\rho\to 0} \rho\cos\varphi\, \rho^{2\alpha} =$$

$$= \lim_{\rho\to 0} \rho \underbrace{\cos\varphi} = \begin{cases} 0, & \text{kada je } 2\alpha+1>0 \\ \text{neodređeno} & \text{kada } 2\alpha+1\le 0 \end{cases}$$

$\cos\varphi$ je uvijek ograničena između -1 i 1 tako da ova vrijednost ne utiče na postojanje limesa

$$2\alpha+1>0 \iff \alpha > -\frac{1}{2}$$

Odgovor: Limes iznad će postojati akko je $\alpha > -\frac{1}{2}$.

#

Ako je $z=z(x,y)$ i $x+y+z=f(x^2+y^2+z^2)$ provjeriti da li je tačna jednakost

$$(y-z)\cdot\frac{\partial z}{\partial x}+(z-x)\frac{\partial z}{\partial y}=x-y.$$

Rj. $z=z(x,y) \Rightarrow$ z je f-ja dvije promjenjive x i y.

$$z=f(x^2+y^2+z^2)-x-y$$

$$t=x^2+y^2+z^2$$

$$s=-x-y$$

$$z=f(t)+s$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial t}\cdot\frac{\partial t}{\partial x}+\frac{\partial s}{\partial x}$$

$$\frac{\partial z}{\partial x}=f'_t\cdot\left(2x+2z\frac{\partial z}{\partial x}\right)-1$$

$$\frac{\partial z}{\partial x}-f'_t\cdot 2z\frac{\partial z}{\partial x}=f'_t\cdot 2x-1$$

$$\frac{\partial z}{\partial x}=\frac{2xf'_t-1}{1-2zf'_t}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial t}\cdot\frac{\partial t}{\partial y}+\frac{\partial s}{\partial y}$$

$$\frac{\partial z}{\partial y}=f'_t\cdot\left(2y+2z\frac{\partial z}{\partial x}\right)-1$$

$$\frac{\partial z}{\partial y}-2zf'_t\frac{\partial z}{\partial x}=2yf'_t-1$$

$$\frac{\partial z}{\partial y}=\frac{2yf'_t-1}{1-2zf'_t}$$

$$(y-z)\cdot\frac{\partial z}{\partial x}+(z-x)\frac{\partial z}{\partial y}=\frac{(y-z)(2xf'_t-1)}{1-2zf'_t}+\frac{(z-x)(2yf'_t-1)}{1-2zf'_t}=$$

$$=\frac{\cancel{2xyf'_t}-y-2xzf'_t+z+2yzf'_t-z\cancel{-2xyf'_t}+x}{1-2zf'_t}=$$

$$=\frac{(x-y)-2xzf'_t+2yzf'_t}{1-2zf'_t}=\frac{(x-y)+2zf'_t(-x+y)}{1-2zf'_t}=$$

$$=\frac{(x-y)(1-2zf'_t)}{1-2zf'_t}=x-y$$

Izračunati integral $\iint_D \sqrt{x^2+y^2}\,dx\,dy$ gdje je

$$D = \{(x,y) \in \mathbb{R}^2 \mid x \le x^2+y^2 \le 2x\}.$$

Rj.

$x = x^2+y^2$

$x^2 - x + y^2 = 0$

$x^2 - 2\cdot x \cdot \frac{1}{2} + \frac{1}{4} + y^2 = \frac{1}{4}$

$(x-\frac{1}{2})^2 + y^2 = (\frac{1}{2})^2$

$x^2+y^2 = 2x$

$x^2 - 2x + y^2 = 0$

$x^2 - 2\cdot x\cdot 1 + 1^2 + y^2 = 1^2$

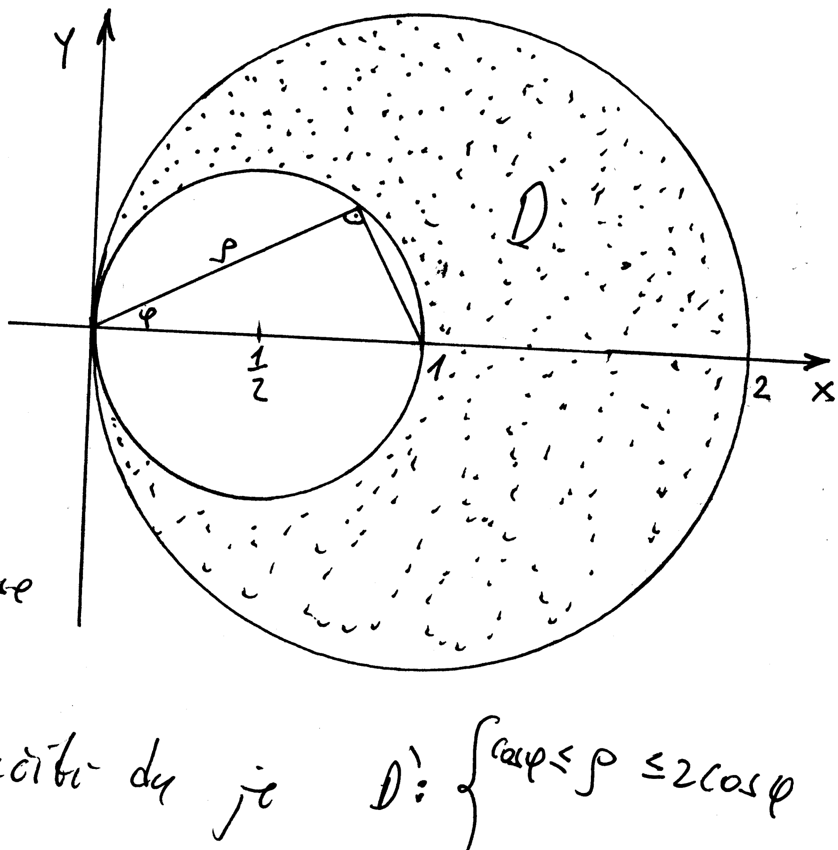
$(x-1)^2 + y^2 = 1$

$\cos\varphi = \frac{\rho}{1}$

$\rho = \cos\varphi$

$\cos\varphi = \frac{\rho}{2}$

Skicirajmo oblast D.



Ako uvedemo polarne koordinate

$x = \rho\cos\varphi$

$y = \rho\sin\varphi$

$dx\,dy = \rho\,d\rho\,d\varphi$

$D \xrightarrow{\text{transfor.}} D'$

unutrašnjost prvog kruga možemo opisati sa $\begin{cases} 0 \le \rho \le \cos\varphi \\ -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \end{cases}$

Iz čega nije teško zaključiti da je $D' : \begin{cases} \cos\varphi \le \rho \le 2\cos\varphi \\ -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \end{cases}$

Kako je $x^2+y^2 = \rho^2$ sad imamo

$$\iint_D \sqrt{x^2+y^2}\,dx\,dy = \left|\begin{matrix}\text{uvodimo}\\ \text{polarne}\\ \text{koordinate}\end{matrix}\right| = \iint_{D'} \rho\rho\,d\rho\,d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_{\cos\varphi}^{2\cos\varphi} \rho^2\,d\rho = \frac{7}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\varphi\,d\varphi$$

$$= \frac{7}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\sin^2\varphi)\,d(\sin\varphi) = \ldots = \frac{28}{9}$$